Communication activity: temporal correlations, clustering, and growth

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Communication via electronic mail represents a form of human dynamics. Embedded in a social network, the communicating partners interact in a complex fashion, where the act of communication is triggered by internal and external influences. Nevertheless, the timing of communication is not completely random—on the contrary, communication is dominated by emergent statistical laws. We recently found long-term correlations in the activity of sending messages in social communities and were able to relate non-trivial growth properties to this type of memory. However, the origins of this persistence are unclear: From a statistical physics point of view long-term correlations can be due to (i) power-law distributed inter-event times (Levy correlations) or (ii) dependencies between the activity at different times. Here we investigate the times when messages are sent in two social communities and find evidences indicating a superposition of both scenarios. We apply stretched exponential fits to the inter-event time distributions which are supported by peak over threshold simulation model reproducing (a) long-term correlations and (b) growth properties. Moreover, we review, confirm, and extend our previous results regarding long-term correlations and the relation to growth processes. Our results suggest a unified framework which also encompasses the inter-event time distributions.

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I. INTRODUCTION
In this study we concentrate on the communication activity of actants in social communities with a special consideration of the timing and investigate three scaling laws of human communication: (i) long-term correlations
in the communication, (ii) growth process of communication or establishing acquaintances, and (iii) inter-event time distribution of subsequent messages. While the first two, long-term correlations and growth processes, have been studied and already put into a framework [1], the third one, inter-event time distributions, has been studied in other publications, such as [2,3], but not yet put in the context of (i) and (ii). In the present paper we review and detail the empirical findings of [1] regarding (i) and (ii), provide further evidence regarding (iii) the inter-event time distributions and propose means to join these three empirical findings into a common framework.

On the one hand, long-term correlations have been found in the dynamics of many physical, technological, and natural systems. They are characterized by a divergent correlation time, i.e. a power-law decaying auto-correlation function (for a review see [4]). Such correlations lead to a pronounced mountain-valley-structure on all scales – comprising indeterministic epochs of small and large values. This type of persistence represents a surprising regularity since it is present in many different data such as DNA-sequences, climatological temperature, human heartbeat, etc. [5–7]. Apart from the volatility of stocks, these long-term correlations have rarely been described in a human social context.

On the other hand, studying economic data surprising growth patterns have been identified [8], which seem to be abundant in systems comprising growth like features. Considering the units of a system of interest, their logarithmic growth rate between two time steps was assumed to be independent of the size of the units, at least in the case of companies. In contrast, [8] showed that the average growth rate depends on the size of the units following a power-law, whereas large units comprise smaller growth rates. In addition, the standard deviation of the growth rates as well follows a power-law, although in general with a different exponent. The latter is related to correlations in the system [1].

In our context, the term ‘inter-event times’ refers to times between successive messages of individual persons. The distributions of such inter-event times have been found to be rather broad. If many short intervals are separated by few long ones, the activity as messages per unit time comprises persistence, i.e. epochs of small and large activity. Since such distributions have been described with power-laws, we wish to test two possible scenarios. (i) Are the long-term correlations found in the communication activity [1] results from Levy type distributions, i.e. correlations which are only due to a power-law inter-event time distribution (with exponents in the specific range) [8]? In the second scenario, (ii) this is not the case, but the activity comprises ‘real’ correlations, i.e. the inter-event times distributions do not follow a power-law, but the communication activity is temporally not independent, but long-term correlated.

This paper is organized as follows. In the data section we briefly describe the online communities from where the communication activity was logged and what the data includes. In Sec. II we present the results on long-term correlations, inter-event time distributions, growth processes, and other correlations. Simulations of a model that reproduces correlation and growth features is proposed in Sec. III. Finally, the conclusions are drawn, followed by an Appendix providing some analytical considerations.

II. DATA

We analyze the dynamics of sending messages in two Internet communities in search of statistical laws of human communication activity. The first online community (OC1) is mainly used by the group of men who have sex with men (MSM) [99]. The data consists of over 80,000 members and more than 12.5 million messages sent during 63 days. The target group of the second online community (OC2) is teenagers [10,12]. The data covers 492 days of activity with more than 500,000 messages sent among almost 30,000 members. Both web-sites are also used for social interactions in general. All data are completely anonymous, lack any message content and consist only of the time when the messages are sent and identification numbers of the senders and receivers.

Similarly to other online communities, the members

[99] The study of the de-identified MSM dating site network data was approved by the Regional Ethical Review board in Stockholm, record 2005/5:3.
can log in and meet virtually. There are different ways of interacting in these communities. Common among most of such online communities is the possibility to choose favorites, i.e. a list of other members, that a person somehow feels committed to. In addition, the platforms offer the possibility to join groups and discuss with other members about specific topics. We focus on the messages sent among the members. These messages are similar to e-mails but have the advantage that they are sent within a closed community where there are no messages coming from or going outside.

Figure 1 shows with vertical lines (a) the sending activity and (b) receiving (passive) of a typical member of OC1. The messages represent a point process [13]. The activity and (b) receiving (passive) of a typical member of a closed community where there are no messages coming

Figure 2: Temporal correlations in the daily communication activity of OC1 members. The panels show DFA2 fluctuation functions versus the time scale $\Delta t$ averaged conditional to the final number of messages $M$. (a,c) Sending; (b,d) receiving; (a,b) original; and (c,d) shuf
corresponding cumulative number of messages, $\mu(t)$. The so called counting process. The panels (e) and (f) show the aggregated number of messages, $\mu(t)$, which is also known as sequence of counts, in our case number of messages per day (OC1) or per week (OC2).

III. RESULTS

A. Long-term correlations

First we study the activity records, $\mu(t)$, with respect to long-term correlations. Figure 2 shows for OC1 fluctuation functions, $F_{DFA2}(\Delta t)$, obtained from second order Detrended Fluctuation Analysis (DFA2) [5,7] (linear detrending of $\mu(t)$, see Appendix A). Long-term correlations are characterized by a power-law decaying autocorrelation function. In this case, the fluctuation function provided by DFA scales as $F(\Delta t) \sim (\Delta t)^H$ with $1/2 < H < 1$ (larger exponents correspond to more pronounced long-term correlations). For uncorrelated or short-term correlated records the asymptotic fluctuation exponent is $H = 1/2$. The $F(\Delta t)$ have been calculated for sequences of messages per day, namely individually for each member. The Figure shows the cases of sending activity and passive receiving for original data and shuffled data. Panel (a) corresponds to sending and one can see that after some curvature on short time scales, which is an artifact of DFA2 [14], the $F(\Delta t)$ curves follow power-laws with exponents close to $1/2$ for low active members and up to $3/4$ for the very active members. In contrast, for the case of shuffled data all fluctuation functions follow asymptotic straight lines with exponents $1/2$ [panel (b)] confirming the lack of correlations. For receiving messages, the corresponding results [panel (c) and (d)] look very similar, which is already indicated by Fig. 1. In the case of OC2 (Fig. 3), for which we have performed the same analysis (but in weekly resolution), we obtain similar results although with somewhat stronger correlations namely fluctuation exponents around 1 which corresponds to $1/f$-noise. One explanation for this difference could be that the OC1 dataset covers a much smaller period of about two months while the OC2 dataset covers more than a year (Fig. 2 is in daily, while Fig. 3 is in a weekly resolution).

We measure the fluctuation exponents by applying least squares fits to $\log F(\Delta t)$ vs. $\log \Delta t$ in the scales $[545]$-5487, 5488-1432, 1433-4033, 4034-10942, 10943-27287, 27288-81042, (from bottom to top). The straight lines in (a,b) correspond to different activity levels: (a,b) original; and (c,d) shuffled. The different curves in each panel correspond to different activity levels: $M = 1-2, 3-7, 8-20, 21-54, 55-148, 149-403, 404-1096, 1097-2980, 2981-8103$, $8104-22026$ (from bottom to top). The straight lines in (a,b) correspond to the exponents $H = 0.75$ (top) and $H = 1/2$ (bottom) as well as $H = 1/2$ (larger exponents correspond to more

3−7
8−20

FIG. 3: Temporal correlations in the weekly communication activity of OC2 members. Analogous to Fig. 2. The top straight lines in (a,b) correspond to the exponents $H = 1$.
For receiving messages, Fig. 4(b), we find almost identical results. The error bars in Fig. 4 were calculated by subdividing the groups of different activity level. The size of the error bars is simply the standard deviation of the corresponding exponents.

Figure 4 also exhibits the fluctuation exponents when we shuffle the data but preserve the distribution of the inter-event times, see Sec. III B. For members with low activity, up to $M \approx 100$, the fluctuation exponents are almost identical to the ones estimated for the (unshuffled) original data. However, for members with high activity, $M > 100$, the fluctuation exponents for the shuffled data preserving the inter-event times deviate from the fluctuation exponents of the original data and are clearly located below them. If the long-term correlations would be only due to ‘real’ correlations then the exponents of the shuffled data should be $1/2$, which is not the case. This result suggests that the distribution of inter-event times plays an important role. Thus, both processes contribute to the long-term persistence exhibited in Fig. 4. Levy type correlations and ‘real’ long-term correlations.

The estimated fluctuation exponents obtained from Fig. 3 for OC2 are displayed in Fig. 5. Qualitatively, we obtain a similar picture as for OC1. However, in contrast to OC1, here the original records achieve larger fluctuation exponents up to $0.91 \pm 0.04$ (sending, disregarding the last points which carry large error-bars). As mentioned before, a possible reason for these different maximum exponents could be that in the case of OC2 the data covers a much longer period of data acquisition, and possible non-stationarities [15]. In OC1, the members might not have had enough time to exhibit the full extent of their persistence. Another difference between OC1 and OC2 is that in the latter the exponents of the shuffled data preserving the distribution of inter-event times also reach high values, almost as high as for the original data, and do not really drop for very active members.

Indeed, similar behavior of long-term correlations have been found in traded values of stocks and e-mail communication [16, 17], where the fluctuation exponent increases in an analogous way with the mean trading activity of the corresponding stock or with the average number of e-mails.

In Fig. 5, we compare for sending in OC2 the fluctuation functions in daily resolution [Fig. 5(a)] and weekly resolution [Fig. 5(b)], which are identical to those of Fig. 4(a). In order to match the scales, we have shifted the curves in Fig. 5(b) along the $\Delta t$-axis. Naturally, in daily resolution, the fluctuation functions cover more scales. Again, the curvature on short time scales appear, but the asymptotic scaling is in both cases the same, namely no correlations in the case of least active members and strong long-term correlations with fluctuation exponents around 1 for the most active members. Moreover, for the latter, the fluctuation functions exhibit an increase from small slopes on short time scales to larger slopes on large scales. This indicates that the long-term correlations do not vanish after certain scale, but the opposite, the long-
term correlations become stronger. Note, that we use weekly resolution in order to cope with possible weekly oscillations [3, 18, 19].

For human related data, long-term persistence has been reported for physiological records [7, 20], written language [21], or for records generated by collective behavior such as finance and economy [22, 23], Ethernet traffic [25], as well as highway traffic [26]. There are also indications of long-term correlations in human brain activity [27, 28] and human motor activity [29].

An interesting question arises, why the fluctuation exponent (in Figs. 3 and 5) depends on the activity level [27, 28] and human motor activity [29].

In Fig. 6 we study for OC1 the long-term correlations in the daily amount of messages on directed links in OC1. (a) DFA2 fluctuation functions versus the time scale ∆t averaged conditional to the final number of messages of each link. The different curves correspond to different activity levels: \( M = 1-2, 3-7, 8-20, 21-54, 55-148, 149-403, 404-1096, 1097-2980 \) (from bottom to top). The dotted lines correspond to the exponents \( H = 0.75 \) (top) and \( H = 1/2 \) (bottom). (b) Fluctuation exponents of the link activity of OC1 members. The DFA2 fluctuation exponent \( H_{OC1} \) is plotted as a function of the activity level \( M \) obtained from (a). The exponents were obtained in the range 10 ≤ s ≤ 63 days. The activity along directed links comprise similar long-term correlations as the total activity of individual members to all of their acquaintances.

B. Inter-event time distributions and clustering in the communication activity

The timing of human communication activity has been found to comprise bursts where many events occur in relatively short periods which are separated by long periods with few or no events at all. Such patterns can be quantified by the so called 'inter-event times', i.e. the times, \( dt \), between successive messages of individuals. For e-mail communication it has been argued that their probability density follows a power-law,

\[
p(dt) \sim (dt)^{-\delta}
\]

with exponent \( \delta \approx 1 \) [2, 31]. As an origin for such heavy tails in human dynamics a queuing model has been suggested [2] according to which each individual performs tasks from a priority list. It has been confirmed that such a process can reproduce bursts of activity, see e.g. [32, 33]. In contrast, analyzing the same e-mail data, a log-normal distribution has been found to be more appropriate to describe the inter-event time distribution.
We would like to remark that fitting fat tailed distributions is disputed. There is neither a consensus on a typical functional form nor on a proper fitting technique. Recently, a cascading Poisson process based on daily and weekly cycles has been proposed as origin of slower-than-exponential decays of $p(dt)$.

Regarding memory between the time sequence of $dt$, Goh and Barabási studied different data sets and characterized the inter-event times in terms of a burstiness parameter, which is based on the distribution, and in terms of a memory coefficient, which is the auto-correlation function at lag 1. In addition, the authors locate the corresponding data sets in a phase diagram defined by these two quantities. Nevertheless, we would like to note that the quantification of correlations in the $dt$ can be hindered by noise.

In this section we study the inter-event times $dt$ between successive messages of individual members, and try to relate their statistics to the persistence described above. The finding of long-term correlations opens the question of the origin of such a persistence pattern in the social communication. From a statistical physics point of view, we speculate on two possible scenarios. The question is whether the finding of an exponent $H > 1/2$ is purely due to a power-law (Levy type) distribution of the time intervals between two messages of the same person (inter-event times) or just from pure correlations or long-term memory in the activity of people. In the first scenario, the intervals between the messages should follow a power-law. Accordingly, the activity pattern comprises many short intervals and few long ones, implying persistent epochs of small and large activity. This fractal-like clustering in the activity can – depending on the exponent – lead to long-term correlations with $H > 1/2$ (see the analogous problem of the origin of long-term correlations in DNA sequences as discussed in [13]). This scenario implies a direct link between the correlations in activity and the distribution of inter-event times which can be obtained analytically (see Appendix B). In the second scenario, the intervals between the messages do not follow a power-law distribution, but the values of the inter-event times are not independent of each other, but comprise long-term persistence. For example, the distribution of inter-event times could be stretched exponential (see recent work on the study of extreme events of climatological records exhibiting long-term correlations [12, 13]) and then the only way to explain long-term correlations in activity is to assume correlations in the inter-event times.

While in the case of Levy type correlations shuffling the inter-event times should not influence the correlation properties, in the case of ‘real’ long-term correlations shuffling the inter-event times should destroy the (asymptotic) long-term correlations since the memory is due to the arrangement of the inter-event times.

Figure 8 shows the probability density, $p(dt)$, of times between messages sent in both online communities. In the case of OC2 a power-law regime of approximately two decades can be seen with an exponent $\delta \approx 1.5$, which differs from the exponent obtained for the e-mail communication mentioned before. In the case of OC1 the scaling is not as good and the exponent seems to be even somewhat larger.

Since we found a dependence of the fluctuation exponent $H$ on the activity level $M$, i.e. the total number of messages each member sends, we suspect that also $\delta$ might depend on $M$. Thus, in Fig. 9 we plot for sending in OC1 (daily resolution) the $p(dt)$ for groups of different activities, i.e. different total number of messages $M$. We find that for the most active members $p(dt)$ decays rather steep, while for the least active members $p(dt)$ decays much slower. Due to the finite size of the data it is not quite clear which functional form the curves follow.
If one assumes a power-law decay then the exponents are roughly in the range $1 \leq \delta \leq 3$.

The analogous results for OC2 are depicted in Fig. 10. Obviously larger values of $dt$ occur because the period of data acquisition is much longer. At the same time the most active members are less active than those of OC1. Nevertheless, qualitatively the curves look similar to those of Fig. 9.

A power-law distribution of inter-event times, Eq. (1), can lead to a type of long-term correlations in activity, without requiring temporal dependencies between the intervals themselves. The long-term persistence properties of this point process can be characterized with the fluctuation exponent which theoretically depends on $\delta$ according to

$$H_\delta = \begin{cases} 
\frac{\delta}{2} & \text{for } 1 < \delta < 2 \\
2 - \frac{\delta}{2} & \text{for } 2 < \delta < 3 \\
\frac{1}{2} & \text{else}
\end{cases}$$

(2)

see Fig. 11 and Appendix B.

Applying least squares fits (in the straight range) to the $p(dt)$ for sending in OC1 and OC2, respectively (Fig. 9 and 10) we obtain values for $\delta$ as a function of the actual activity level $M$ and get the corresponding fluctuation exponents, $H_\delta$, as expected from Eq. (2). We would like to note that the curves in Fig. 9 and 10 are not always nice straight lines leading to large uncertainty regarding the estimated values of $\delta$. Figure 12 depicts the fluctuation exponents $H_\delta$ in comparison with the values obtained from DFA (Fig. 4(a)). It can be seen that for OC1, $H_\delta > H$ in a wide range, in particular for intermediate $M$ and $H_\delta < H$ in the large $M$ regime. The values are not compatible, showing that for OC1 the correlations cannot be explained only with the distribution of the inter-event times, with the consequence that Levy type correlations might not be the dominant process. In addition, the $H$ for the shuffled records (where the inter-event time distribution is preserved) become smaller and deviate for large $M$ from the $H$ of the original data, supporting the existence of 'real' correlation in contrast to only Levy type correlations. The most active members (large $M$) should not comprise long-term correlations according to the estimated $p(dt)$. The fact that their activity is long-term correlated is a sign of 'real' correlations. The inter-event time distribution might be broad, but not broad enough to account for the long-term correlations, as we can conclude from the reduced $H$ in the case of the shuffled records.

In contrast, for OC2, we find $H \approx H_\delta$ for a big part of the $M$ range, showing that in this case the correlations can be due to a power-law distribution of the inter-event times, which is in favor of Levy type correlations. Furthermore, the shuffling preserving the inter-event time distribution also leads to fluctuation exponents that are compatible with the ones obtained from the original data. Thus, for OC2 all three curves are in a reasonable agreement which supports the Levy type correlations. Never-
the average inter-event time of each activity level $M$. The solid lines represent best fits to the measured data, namely (a) 2-parameters fit, Eq. (3), resulting in $a = 10.96$, $b = 0.29$ and (b) 1-parameter fit, Eq. (4), resulting in $\gamma_{se} = 0.15$.

Nevertheless, the findings for both data sets do not agree – contrasting the intuition that the underlying processes for both online communities must be the same. Accordingly, it remains unclear whether the discrepancy between both data sets is due to different properties or due to some measurement problem, such as that the OC1 data covers only 63 days whereas for OC2 it covers the complete lifetime of the community (which might be a reason for non-stationarities).

Another approach to analyze the inter-event time distribution is to collapse the $p(dt)$ by accounting the average inter-event time, $\langle dt \rangle$ of each activity level $M$. The scaled probability densities are obtained by dividing the $dt$-axis by $\langle dt \rangle$ of the corresponding group and multiplying $p(dt)$ with $\langle dt \rangle$, thus, by plotting $p(dt) \ast \langle dt \rangle$ versus $dt/\langle dt \rangle$.

The collapse scaled curves are shown in Fig. 13 and 14. Indeed, the curves fall on each other (similar collapse of message data have been shown before, such as in Fig. 11). In some cases, we see a hump in the collapsed data, namely at around 5dt/\langle dt \rangle. We are not sure about its origin. It could be a consequence of fluctuating activity due to weekly oscillations. Disregarding this hump, the functional form of these scaled probability densities is not obvious. Arguing that each set in Fig. 9 with different slope piecewise contributes to a new curve, that is not a straight line, in the scaled probability densities, we elaborate two versions of a stretched exponential and fit them to the data. The stretched exponential is an interesting candidate since it represents a hybrid type between exponential and power-law.

The first fitting function is

$$p(dt) = b \frac{a}{R_M} \left( \frac{a}{R_M} \langle dt \rangle \right)^{b-1} e^{-\left( \frac{a}{R_M} \langle dt \rangle \right)^b},$$

where $a$ and $b$ are parameters and $R_M = \langle dt \rangle$ is the conditional average inter-event time. For $0 < b < 1$ the functional form is between exponential and a power-law (with exponent $\delta = 1$). The second fitting function is

$$p_M(dt) = \frac{a_\gamma}{R_M} e^{-b_\gamma \left( \frac{dt}{\langle dt \rangle} \right)^{\gamma_{se}}},$$

where $\gamma_{se}$ is a fitting parameter. The factors $a_\gamma$ and $b_\gamma$ are given by Eq. (5):

$$a_\gamma = \frac{\gamma_{se} \Gamma(2/\gamma_{se})}{\Gamma^2(1/\gamma_{se})} \quad b_\gamma = \left( \frac{\Gamma(2/\gamma_{se})}{\gamma_{se} \Gamma(1 + 1/\gamma_{se})} \right)^{\gamma_{se}}$$

and represent normalization conditions. Eq. (4) is motivated by findings regarding our model of peaks over thresholds of time series (see Sec. IV). Also, it has been shown that for long-term persistent records, the intervals between extreme events above a certain threshold follow a stretched exponential probability density according to Eq. (4). Thus, long-term memory represents a natural mechanism for the pronounced clustering of extreme events. This phenomenon is also reflected in climatological records. Particularly relevant is the finding that $\gamma_{se} = \gamma$ [42, 49], i.e. the long-term correlation exponent is the same as the exponent of the stretched exponential.

In Sec. IV this feature is used for model simulations.

In Fig. 13(a) we use the 2-parameter fit form, Eq. (3), and obtain the exponent $b \approx 0.29$. Applying the 1-parameter fit form, Eq. (4), we find the exponent $\gamma_{se} \approx 0.15$ [Fig. 13(b)]. In Fig. 14(a) we obtained a maximum fluctuation exponents $H_{OC1} \simeq 0.75$ which corresponds to $\gamma_{OC1} = 0.5$, Eq. (8). Thus, this discrepancy either does not support 'real' long-term correlations (in contrast to the Levy type) or the stretched exponential with $\gamma_{se} = \gamma$ (see Appendix IV) is not suitable. Accordingly, these contradicting findings suggest that the correlations found in the activity are due to both, Levy type distributions and 'real' long-term correlations in the inter-event times.

The stretched exponential fits for OC2 are displayed in Fig. 14. For the 2-parameter fit we find $b \approx 0.30$ [Fig. 14(a)] and for the 1-parameter fit $\gamma_{se} \approx 0.14$. Our
finding $H_{OC2} \simeq 0.91$ ($\gamma_{OC2} = 0.18$) is more in agreement with the value of $\gamma_{oc} \approx 0.14$ supporting the concept of ‘real’ correlations. However, this contrasts with the good agreement of Fig. 12(b) which supports Levy correlations in the case of OC2.

In summary, we find evidence for both, ‘real’ correlations and Levy type correlations, and conclude that from these two sets of message data, we do not get an unambiguous picture and further research is required to clarify the nature of the long-term correlations in the message activity. It would be desirable to study more message communication data from other online communities, e-mail communication, and letter post [40, 50]. A possible methodological extension could be to investigate multifractal properties [51, 53] of the daily messages records, $\mu(t)$. While a power-law distribution of inter-event times according to Levy type correlations implies a broad multifractal spectrum [51, 53, 54], for interdependent inter-event times (‘real’ correlations scenario) the multifractal spectrum is expected to be different and more narrow (unless it contains additional multifractal features). Another open question requiring further studies emerges from the finding of log-normal inter-event time distributions [54]. In this case it is not clear, how this third scenario is related to long-term correlations and how it could be integrated in the framework presented so far.

C. Growth process

1. Growth in the number of messages

As suggested in [1] we also analyze the growth properties of the message activity. This concept is borrowed from econophysics, where the growth of companies has been found to exhibit non-trivial scaling laws [8, 59] and at the same time represents a generalized Gibrat’s law (GGL) [1]. In the present study, each member is considered as a unit and the number of messages sent or received since the beginning of data acquisition represents its size. We analyze the growth in the number of messages in analogy to other systems such as the growth of companies [8, 59] or the growth of cities [60], see also [61, 63]. (i) The members of a community represent a population similar to the population of a country. (ii) The number of members fluctuates and typically grows similar to the size of cities. (iii) The activity or number of links of individuals fluctuates and grows similar to the size of cities.

The cumulative number, $m_j(t)$, expresses how many messages have been sent by a certain member $j$ up to a given time $t$ [for a better readability we will not write the index $j$ explicitly, $m(t)$]. We consider the evolution of $m(t)$ between times $t_0$ and $t_1$ within the period of data acquisition $T$ ($t_0 < t_1 \leq T$) as a growth process, where each member exhibits a specific growth rate $r_j$ ($r$ for short notation):

$$r = \ln \frac{m_1}{m_0}$$

where $m_0 \equiv m(t_0)$ and $m_1 \equiv m(t_1)$ are the number of messages sent until $t_0$ and $t_1$, respectively, by every member. To characterize the dynamics of the activity, we consider two measures. (i) The conditional average growth rate, $\langle r(m_0) \rangle$, quantifies the average growth of the number of messages sent by the members between $t_0$ and $t_1$ depending on the initial number of messages, $m_0$. In other words, we consider the average growth rate of only those members that have sent $m_0$ messages until $t_0$. (ii) The conditional standard deviation of the growth rate for those members that have sent $m_0$ messages until $t_0$.

$$\sigma(m_0) \equiv \sqrt{\langle (r(m_0) - \langle r(m_0) \rangle)^2 \rangle}$$

expresses the statistical spread or fluctuation of growth among the members depending on $m_0$. Both quantities are relevant in the context of Gibrat’s law in economics [52, 57] which proposes a proportionate growth process entailing the assumption that the average and the standard deviation of the growth rate of a given economic indicator are constant and independent of the specific indicator value. That is, both $\langle r(m_0) \rangle$ and $\sigma(m_0)$ are independent of $m_0$.

For OC1 the average growth rate and the standard deviation as a function of the initial number of messages, $m_0$, are shown in Fig. 15. While we fix $t_1 = T$ in order not
to disregard any data, we vary $t_0$ within $0 < t_0 < T$. For sending [Fig. 15(a+c)] and receiving [Fig. 15(b+d)] we find very similar results. The conditional average growth rate is almost constant and only decreases slightly

$$\langle r(m_0) \rangle \sim m_0^{-\alpha}$$

with an exponent $\alpha \approx 0.05$. This means that members with many messages in average increase their number of messages almost with the same rate as members with few messages. In contrast, the conditional standard deviation clearly decreases with increasing $m_0$. The straight lines in the double-logarithmic representation correspond to

$$\sigma(m_0) \sim m_0^{-\beta}$$

where the exponents $\beta$ approximately is in the range $0.2 < \beta < 0.25$. We find that $t_0 = T/2$ is optimal in terms statistics [1]. In this case we obtain $\beta_{OC1} = 0.22 \pm 0.01$ for sending. This means, although the average growth rate almost does not depend on $m_0$, the conditional standard deviation of the growth of members with many messages is smaller than the one of members with few messages. Accordingly, the activity of sending messages and the passivity of receiving messages of active members exhibits less fluctuations and is therefore better predictable.

In order to better understand these scaling laws, Eq. (5) and (9), we next shuffle the message data. This is done by randomizing the instants at which the messages are sent. We keep all messages and all time stamps but shuffle the associations between times and messages among the entire set of messages in the period of data acquisition [1]. An example of $m(t)$ for the shuffled data is shown in Fig. 1 and one can clearly see that compared to the original data, which exhibits strong variation of growth, the shuffled data leads to a almost linearly increasing $m(t)$.

Figure 16 shows the same quantities as Fig. 15 but for the shuffled data. For any $t_0$, sending or receiving, the average growth rates are constant and do not depend on $m_0$ ($\alpha = 0$). In particular, the conditional standard deviation now exhibits a steeper decay with $m_0$, namely following a power-law, Eq. (9), with exponent $\beta_{rnd} = 1/2$. We conclude, that this difference must be due to the temporal correlations found in Sec. III A, and that the exponent $\beta$ must be related to the exponent $H$ [1], see Appendix C.

We perform the same analysis for OC2, Figs. 17+18. For the original data, again, we obtain almost constant average growth rate, not depending on the initial number of messages $m_0$ [Fig. 17(a+b)]. The conditional standard deviation of the growth rates decays approximately following a power-law, Eq. (9), with exponents $\beta \approx 0.2$ as indicated by the dotted lines in Fig. 17(c+d). In the optimal case $t_0 = T/2$ we measure $\beta_{OC2} = 0.17 \pm 0.03$. Figure 18 shows the results for the shuffled data. Again, we find apart from small deviations for small $m_0$ straight lines with $\alpha_{rnd} = 0$ and $\beta_{rnd} = 1/2$, consistent with the findings for OC1. For OC2 the curves are more noisy compared to OC1, which is due to the smaller amount of data available from OC2.

Although the web-sites are used by different member populations, the power-laws and the obtained exponents are quite similar. The exponents are also close to those reported for growth in economic systems such as firms (0.15 – 0.18, [8]) and countries (0.15 ± 0.03, [9]), research and development expenditures at universities (0.25, [5]), voluntary organizations (0.19, [10]), scien-
is related to the maximum value of $H$ slopes in the Figs. 15 and 17. However, the simulations in $M_\beta$ bers (Fig. 4 and 5) should also imply a dependence of $versus foregoing number of messages for the shuffled data.

FIG. 18: Average and standard deviation of the growth rate $H$ on the activity level $\beta$.

It has been shown that the fluctuation exponent $H$ and the growth fluctuation exponent $\beta$ are related via

$$\beta = 1 - H,$$  \hspace{1cm} (10)

see Appendix C. Equation (10) is a scaling law formalizing the relation between growth and long-term correlations in the activity which is supported by our data. For OC1 we measured $\beta_{OC1} \approx 0.22$ yielding $H_{OC1} \approx 0.78$ from Eq. (10), which is in approximate agreement with the (maximum) exponent we obtained by direct measurements for OC1 [$H = 0.75 \pm 0.05$ from Fig. 14(a)]. For OC2 we obtained $\beta_{OC2} \approx 0.17$ and therefore $H_{OC2} \approx 0.83$ through Eq. (10) which is not too far from the (maximum) exponent found by direct measurements for OC2 [$H = 0.91 \pm 0.04$, Fig. 5(a)]. According to Eq. (10), the original Gibrat’s law ($\beta_G = 0$) corresponds to very strong long-term correlations with $H_G = 1$. This is the case when the activity on all time scales exhibits equally strong correlations. In contrast, $\beta_{rnd} = 1/2$ represents completely random activity ($H_{rnd} = 1/2$).

One may argue that the dependence of the fluctuation exponent $H$ on the activity level $M$ of the members (Fig. 4 and 5) should also imply a dependence of $\beta$ on $M$ and accordingly (due to $m_0 \sim M$) lead to changing slopes in the Figs. 13 and 17. However, the simulations in Sec. XV indicate that the growth fluctuation exponent $\beta$ is related to the maximum value of $H$ and that in the case of the less active members their few message do not provide enough information for a correct measurement of the true long-term correlations.

Due to a lack of sufficiently high temporal resolution of economic data, long-term correlations in the growth could not be found. Only by using message data and making the analogy of growth, it is possible to test the relation. Thus, Eq. (10) represents a missing link between growth fluctuations and long-term correlations and accordingly connects both scientific communities who studied these topics separately. In addition, it contributes to understanding the origin of the non-trivial scaling of Eq. 9 with exponents $0 < \beta < 1/2$.

In the case of companies, the distribution of growth rates has also been studied. It has been found that the distribution density follows [6]:

$$p(r|m_0) = \frac{1}{\sqrt{2\pi\sigma(m_0)}} \exp \left( -\frac{r - \langle r(m_0) \rangle}{\sigma(m_0)} \right)$$  \hspace{1cm} (11)

Next we analyze, how the growth rates $r$ are distributed in the case of our social activity data. First we need to point out that in contrast to growth of companies, our growth can never shrink. The members cannot lose messages, the number $m(t)$ either increases or remains the same. Accordingly, in our case $r \geq 0$.

Figure 17 shows $p(r|m_0)$ for OC1 where the values are scaled to collapse according to Eq. (11). In order to have reasonable statistics, we define the condition $m_0$ in rather wide ranges, namely according to the decimal logarithm. For sending [Fig. 17(a)] and receiving [Fig. 17(b)] messages the scaled probability densities collapse rather well, but although the decay seems to be exponential, the slope does not follow Eq. (11) and the prefactor $\sqrt{2}$ might be different.

In the case of OC2 this disagreement becomes more clear, see Fig. 19. For sending [Fig. 19(a)] and receiving [Fig. 19(b)], the $p(r|m_0)$ collapse more or less but do not exhibit the slope expected from Eq. (11). Thus, a
modification of Eq. (11) could lead to a better description of the growth rate distributions.

2. Growth in the degree

Online communities represent a form of social networks. The communication activity via messages provides information on the ties between the members. While above, we were mainly investigating the dynamics of the individuals, next we characterize the growth dynamics of members in the networks. Thus, we consider the members as nodes and the links to be present whenever there is at least one message between two nodes. In general, messages are either being sent establishing a new relation between the corresponding pair of nodes or they follow an already existing link [69]. Starting from a set of isolated nodes we connect them by adding a directed link, when the first message is sent between any pair of members. Thus, we study the growth of the degree, e.g. the links to other nodes that a node has, see also [70]. In the directed network there are out-degree (number of different acquaintances to whom a member sends) and in-degree (number of different acquaintances from whom a member receives messages). It is important to note that in the case of OC2, the period of data acquisition covers the whole lifetime of the online community. In contrast, in the case of OC1, only a limited time window is analyzed. This means, when in OC2 a link appears for the first time it is really new. In contrast, in OC1, when a link appears for the first time, it can be that a link was established earlier. However, this does not seem to have a strong influence on the results.

Figure 21 shows the results for the growth in the degree (OC1), which is defined as before,

\[ r_k = \ln \frac{k_1}{k_0}, \]

where \( k_0 = k(t_0) \) and \( k_1 = k(t_1) \) are the degrees of the specific node at time \( t_0 \) and \( t_1 \), respectively. As for the messages, the average growth rate decreases slowly with \( k_0 \), following a power-law

\[ \langle r_k(k_0) \rangle \sim k_0^{-\alpha_k}, \]

with \( \alpha_k \approx 0.05 \), for in-going as well as out-going links. The conditional standard deviation also decreases as a power-law

\[ \sigma_k(k_0) \sim k_0^{-\beta_k}, \]

with a steeper exponent approximately \( 0.2 < \beta_k < 0.25 \). If we choose \( t_0 = T/2 \), we obtain \( \beta_{k,OC1} = 0.22 \pm 0.02 \) for out-going links. The growth dynamics of sending messages and out-going links is very similar, compare Fig. 21 with Fig. [15].

In the case of OC2, there is less data and the results in Fig. 22 fluctuate much more. The average growth rate is fairly constant and possibly decaying with an exponent \( \alpha_k \approx 0.05 \) (at least for the in-degree). For the conditional standard deviation we see \( 0.2 < \beta_k < 0.25 \). For \( t_0 = T/2 \) and out-degree we obtain \( \beta_{k,OC2} = 0.17 \pm 0.08 \). We conclude, that since in the number of messages the growth fluctuation exponent \( \beta \) is related to the fluctuation exponent \( H \), also the degree growth fluctuations with exponent \( \beta_k \) must be governed by long-term correlations. This is also supported by correlations between sending activity and out-going links [Sec. [11]].
3. Preferential attachment

In order to better understand the mechanism behind the network growth, we wish to compare our findings with the growth properties of a network model. In general, there is considerable interest in the understanding of the dynamics of such social networks, as illustrated by the long-standing controversy on the origin of broad distributions in social and other systems [77, 58, 63, 71, 73]. Two paradigms have been invoked: (i) The “rich-get-richer” idea used by Simon in 1955 [74] for various applications. (ii) The models based on optimization strategies as proposed by Mandelbrot [75, 76].

Next we investigate the Barabasi-Albert (BA) Model which is based on preferential attachment (PA) and has been introduced to generate a kind of scale-free networks [77] with power-law degree distribution \( p(k) \) [78, 79]. Essentially, it consists of subsequently adding nodes to the network by linking them to existing nodes which are chosen randomly with a probability proportional to their degree.

We obtain the undirected network and study the degree growth properties by calculating the conditional average growth rate \( \langle r_{BA}(k_0) \rangle \) and the conditional standard deviation \( \sigma_{BA}(k_0) \), Eqs. (13) and (14). The times \( t_0 \) and \( t_1 \) are defined by the number of nodes attached to the network.

Figure 23 shows the results where an average degree \( \langle k \rangle = 20; 50,000 \) nodes in \( t_0 \), and 100,000 nodes in \( t_1 \) were chosen. We find constant average growth rate that does not depend on the initial degree \( k_0 \). The conditional standard deviation is a function of \( k_0 \) and exhibits a power-law decay characterized by Eq. (14) with \( \beta_{BA} = 1/2 \). Therefore, a purely preferential attachment type of growth is not sufficient to describe the type of social network dynamics found before, since additional temporal correlations are involved in the dynamics of establishing acquaintances in the community.

For the BA model it has been shown that the degree of each node grows in time as \( k(t) \sim \left( \frac{t}{t^*} \right)^{b} \), where \( t^* \) is the time when the corresponding node was introduced to the system and \( b = 1/2 \) is the dynamics exponent in growing network models [80]. Accordingly, the growth rate is given by \( r_{BA} = b \ln \frac{t}{t^*} \), which is constant independent of \( k_0 \), in accordance with our numerical findings. Interestingly, an extension of the standard BA model has been proposed [81], see also [82, 83], that takes into account different fitnesses of the nodes to acquiring links involving a distribution of \( b \)-exponents and therefore a distribution of growth rates. This model opens the possibility to relate the distribution of fitness values to the fluctuations in the growth rates, a point that requires further investigation.

The value \( \beta_{BA} = 1/2 \) in Eq. (10) corresponds to \( H = 1/2 \) indicating complete randomness. There is no memory in the system. Since each addition of a new node is completely independent from precedent ones, there cannot be temporal correlations in the activity of adding links as found in Sec. [11C]. In Appendix [1] we motivate \( \beta_{BA} = 1/2 \).

4. Mutual growth in the number of messages

Next we study another variation of growth. Instead of considering the absolute number messages a member sends, we study the difference in the number of messages
compared to any other member, the mutual difference $m^i(t) - m^j(t)$. Thus, the growth rate is defined analogous to Eq. (15)

$$r_x = \ln \frac{m^i - m^j}{m^j - m^i}$$

where there is a growth rate for every pair of members $i$ and $j$. The conditional average growth rate and the corresponding standard deviation is then taken over all possible pairs and the condition is the difference at $t_0$, $m^i_0 - m^j_0 = m^i(t_0) - m^j(t_0)$, providing the quantities $(r_x(m^i_0 - m^j_0))$ and $\sigma(m^i_0 - m^j_0)$.

The results for sending in OC1 are shown in Fig. 24. Apart from a small decrease up to $m^i_0 - m^j_0 \approx 50$, the average growth rate is constant [Fig. 24(a)]. The conditional standard deviation asymptotically follows a slope $\beta_x \approx 0.3$ with deviations to small exponents for small $m^i_0 - m^j_0$. In the case of the shuffled data [Fig. 24(b)], as expected, the average growth rate is constant while the standard deviation decreases steeper than for the original data, namely with $\beta_{x,\text{shuf}} \approx 1/2$, although not with a nice straight line. Nevertheless, we conclude that the scaling of the standard deviation in Fig. 24(a) must be due to temporal correlations between the members. The growth of the difference between their number of messages comprises similar scaling as the individual growth as seen, e.g., in Fig. 19.

We conjecture that $\sigma(m^i_0 - m^j_0)$ reflects long-term cross-correlations in analogy to $\sigma(m_0)$ for auto-correlations. However, so far, we are not able to provide further evidence for this analogy and the corresponding relation to $\beta = 1 - H$, Eq. (10), since an appropriate technique for the direct quantification of long-term cross-correlations is lacking.

FIG. 24: Average mutual growth rate and standard deviation versus foregoing difference in the number of messages for sending in OC1. The average (open squares) and standard deviation (filled circles) of the mutual growth rate $r_x$, Eq. (15), are plotted conditional to the initial difference $m^i_0 - m^j_0$, whereas $t_0 = T/2$ and $t_1 = T$. (a) Original data and (b) shuffled data. The dotted line in (a) corresponds to the exponent $\beta_x = 0.3$ and in (b) to $\beta_x = 1/2$.

D. Other correlations

In this section we want to discuss other types of correlations. Figure 25 shows for OC1 the final degree $K$ versus the final number of messages $M$ (final means at $T$, e.g. the end of the period of data acquisition). We find that for both, sending and receiving, the two quantities are correlated with an exponent close to 0.75. Similar relations have also been found for other data [84]. Since the correlations are positive, obviously, those members that send many messages, in average, also have more acquaintance to whom they send, but they know less acquaintance than they would in the case of linear correlations. For receiving, Fig. 25(b), this correlation is very similar.

The number of messages sent versus the number of messages received (for OC1) is displayed in Fig. 26(a). Asymptotically the activity and passivity are linearly related and on average for every sent message there is a received one or vice versa. This, of course, does not mean that every message is replied. However, the less active members in average tend to receive more messages than
they send. For example, those members who send in average one message receive about three. Nevertheless, the more active the members are the more sending and receiving behavior approaches the linear relation. In contrast, for the receiving part, Fig. 26(b), the asymptotic linear relation does not hold. Those members with large out-degree and small in-degree are referred to as spammers.

For OC2 we find similar results in Figs. 27 and 28. The final degree and the final number of messages also scale with an exponent close to 0.75, although for sending there exists some deviations in the most active members [Fig. 27(a)]. Also, the correlations of sending and receiving are linearly correlated, as well as in- and out-degree. However, the most active members again deviate with low receiving part, i.e. both low number of received messages as well as low number of in-degree, Fig. 28. Nevertheless, the results for both data sets are mainly consistent and the power-law relation $K \sim M^{0.75}$ is a remarkable regularity.

**IV. PEAKS OVER THRESHOLD (POT) SIMULATIONS**

Our finding that the activity of sending messages exhibits long-term persistence asserts the existence of an underlying long-term correlated process. This can be understood as an unknown individual state driven by various internal and external stimuli [28, 85–88] increasing the probability to send messages. Generating such a hypothetical long-term correlated internal process ($x_i$), simulated message data can be defined by the instants at which this internal process exceeds a threshold $q$ (peaks over threshold, POT) [42, 45].

More precisely, we consider a sequence ($x_i$) consisting of $N^*$ random numbers that is normalized to zero average ($\langle x \rangle = 0$) and unit standard deviation ($\sigma_x = 1$). Choosing a threshold $q$, at each instant $i$ the probability to send a messages is:

$$p_{\text{snd}} = \begin{cases} 1 & \text{for } x_i > q \\ 0 & \text{for } x_i \leq q \end{cases}.$$  (16)

Thus, the message events are given by the indices $i_{\text{snd}}$ of those random numbers $x_i$ exceeding $q$.

Figure. 29(a) illustrates the procedure. The random numbers are plotted as brown circles and the events exceeding the threshold (orange dashed line) by the green diamonds. The resulting instants are depicted in Fig. 29(b) representing the simulated messages. The threshold approximately predefines the total number of events and accordingly the average inter-event time. Using normal-distributed ($x_i$), the number of events/messages is approximately given by the length $N^*$ and the inverse cumulative distribution function associated with the standard normal distribution (probit-function). Additionally, the random numbers we use are

**FIG. 27:** Correlations between the final degree $K$ and the final number of messages $M$ for OC2. (a) Out-degree and sending messages; (b) in-degree and receiving messages. Analogous to Fig. 26.

**FIG. 28:** Correlations between activity and passivity for OC2. (a) Final number of messages $M$ received and sent; (b) final in- and out-degree. Analogous to Fig. 26.

**FIG. 29:** Illustration of the peaks over threshold data. (a) An underlying and unknown long-term correlated process determines the instantaneous probability of sending messages. Once this state passes certain threshold $q$ (dashed orange line) a messages is sent (green diamonds). (b) Generated instants of messages, (c) with windows for aggregation of messages, such as messages per day. (d) Aggregated record of messages in windows of size $w$, here $w = 10$. 
FIG. 30: Results of numerical simulations. (a) Mean growth rate conditional to the number of events until \( t_0 = N^*/2 \) as obtained from 100,000 long-term correlated records of length \( N^* = 131,072 \) with variable imposed fluctuation exponent \( H_{\text{imp}} \) between 1/2 and 0.9 and random threshold \( q \) between 1.0 and 6.0. (b) As before but standard deviation conditional to the number of events. The solid lines represent power-laws with exponents \( \beta \) expected from the imposed long-term correlations according to Eq. (10). (c) Long-term correlations in the sequences of aggregated peaks over threshold. For every threshold \( q \) between 1.0 (violet) and 4.5 (black) 100 normalized records of length \( N^* = 4,194,304 \) have been created with \( H_{\text{imp}} = 0.9 \). The events are aggregated in windows of size \( w = 100 \). The panel shows the averaged DFA2 fluctuation functions. (d) DFA fluctuation exponents on the scales \( 1,000 \leq s \leq 10,000 \), as a function of the total number of events.

long-term correlated with variable fluctuation exponent. We impose these auto-correlations using Fourier Filtering Method (14, 59). Next we want to show that this process reproduces the scaling of e.g. Fig. 15, Eq. (9), as well as the variable long-term correlations in the activity of the members (e.g. Figs. 2 and 4).

For testing this process we create 100,000 independent long-term correlated records \( (x_i) \) of length \( N^* = 131,072 \), impose fluctuation exponent \( H_{\text{imp}} \) and choose for each one a random threshold \( q \) between 1 and 6, each representing a sender. Extracting the peaks over threshold, we obtain the events and determine for each record/member the growth in the number of events/messages between \( N^*/2 \) and \( N^* \). This is, for each record/member we determine the numbers of events/messages \( m_0 \) until \( t_0 = t_{i=N^*/2} \) as well as \( m_1 \) until \( t_1 = t_{i=N^*} \) and calculate the growth rate according to Eq. (9). We then calculate the conditional average \( \langle r(m_0) \rangle \) and the conditional standard deviation \( \sigma(m_0) \) where the values of \( m_0 \) are binned logarithmically. The quantities are plotted in Fig. 20 (a) and (b), while in panel (b) we include slopes expected from \( \beta = 1 - H \), Eq. (10), see Appendix C. We find that the numerical results reasonably agree with the prediction (solid lines). Qualitatively, these results agree with those found for the original message data (Sec. III C).

The fluctuation functions can be studied in the same way. As described in Sec. III A we find long-term correlations in the message data studying the messages per day or per week. On the basis of the above explained simulated message data, we analyze the generated message data in an analogous way. For each threshold \( q = 1.0, 1.5, \ldots 4.0, 4.5 \) we create 100 long-term correlated records of length \( N^* = 4,194,304 \) with imposed fluctuation exponent \( H_{\text{imp}} = 0.9 \), extract the simulated message events, and aggregate them in non-overlapping windows of size \( w = 100 \). This is, tiling \( N^* \) in segments of size \( w \) and counting the number of events occurring in each segment [Fig. 29 (c) and (d)]. The obtained aggregated records represent the analogous of messages per day or per week and are analyzed with DFA averaging the fluctuation functions among those configurations with the same threshold and thus similar number of total events. The corresponding results are shown in Fig 29 (c) and (d). We obtain very similar results as in the original data. We find vanishing correlations for the sequences with few events (large \( q \)) and pronounced long-term correlations for the cases of many events (small \( q \)), while the maximum fluctuation exponent corresponds to the chosen \( H_{\text{imp}} \). This can be understood by the fact that for \( q \) close to zero the sequence of number of events per window converges to the aggregated sequence of 0 or 1 (for \( x < 0 \) or \( x > 0 \)) reflecting the same long-term correlation properties as the original record. For a large threshold \( q \) too few events occur to measure the correct long-term correlations, e.g. the true scaling only turns out on larger unaccessible time scales.

Although the suggested process does not really reveal the origin of the long-term correlated patchy behavior, it supports the framework for the evidences \([\sigma(m_0) \text{ and DFA}], \) reducing the presented findings to a simpler process. Consistently, an uncorrelated, completely random, underlying process recovers Poisson statistics and therefore \( \beta_{\text{rnd}} = 1/2 \) for the growth fluctuations as well as uncorrelated message activity \( (H_{\text{rnd}} = 1/2) \).

We do not need to study the inter-event time distributions of this peaks over threshold approach since it has been shown before [42, 43] that they follow a stretched exponential, Eq. (1). Accordingly, by assuming stretched exponential distributed inter-event times in the community data, this simulation model favors the scenario of ‘real’ correlations (in contrast to Levy type correlations, see Sec. III B), and provides consistent results regarding the relation of the exponents \( \beta \) and \( H \), Eq. (10).

V. CONCLUSIONS

We have studied social communication activity in two online communities with respect to long-term correlations – whereas we address the question of Levy type
or 'real' correlations. In addition, we have investigated the system in terms of growth processes and propose a relation between them. We also studied other correlations, such as between the number of messages and the degree (exponent close to 3/4) as well as sending and receiving (linear relation).

Our work reviews and further supports previous empirical findings [1], extending them by some aspects: We propose two scenarios relating long-term correlations and inter-event time distributions between the successive messages of the individual members. While in the case of Levy type correlations the inter-event times need to be power-law distributed, 'real' long-term correlations are independent of the distributions, since they are due to interdependencies in the activities. Applying shuffling as well as power-law or stretched exponential fits, we find evidences for both scenarios, which are also in agreement with the framework relating long-term correlations and growth fluctuations, Eq. (10). The obtained exponents are summarized in Table I. Extending [1] we also analyze the passivity of receiving messages and find very similar results as for the activity of sending messages. This is due to strong correlations between individual sending and receiving, i.e. most of the messages are somehow replied sooner or later.

Further we find long-term correlations on the basis of links (directed between two members) and not on nodes (members). This means already the communications between two individuals comprises long-term persistence. Regarding the growth in the number of messages as suggested in [1], we extend the previous results by those for receiving. Consistent with long-term correlations, the growth properties of receiving are very similar to those of individuals could be understood as a superposition of many such cascades.

Finally, we propose simulations to reproduce the long-term correlations and growth properties. Basically it consists of generating long-term correlated sequences and defining a threshold. All values of those sequences above the threshold represent a message event. We show that then the correlation and growth features are in agreement with those of the original data, being determined by the imposed fluctuation exponent. The simulations confirm the relation $\beta = 1 - H$, Eq. (10).

We believe that this work gives some insight into the phenomenon of human dynamics in terms of communication via messages. Nevertheless, it also opens perspectives for further research activities. In particular, the origin of the long-term persistence in the communication remains an important question. (i) From a statistical physics point of view, two scenarios were proposed (Levy or 'real' long-term correlations). An option to shed light on this problem could be to analyze multifractal properties of the message sending activity. The two processes are expected to comprise very different multifractal signatures. At the same time one could argue about a third scenario combining both, i.e. power-law distributed inter-event times that are at the same time temporally correlated/interdependent, see also [41]. (ii) From a psychological/sociological point of view one may argue where the persistence is originated. Is it purely due to a state of mind [28], solipsistic, emerging from moods, or is it due to social effects, i.e. that the dynamics in the social network induces persistent fluctuations, such as cascades of replies? An example of the latter case could be that a group of friends tries to make an appointment and therefore send many subsequent messages in a relatively short time. After agreeing, the communication activity among the group drops. The activity patterns of individuals could be understood as a superposition of many such cascades.

<table>
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TABLE I: Overview of the obtained exponents.
Appendix A: Methods

Statistical dependencies between the values of a record $\mu(t)$ with $t = 1, \ldots, T$ can be characterized by the auto-correlation function

$$C(\Delta t) = \frac{1}{\sigma^2_{\mu}(T-\Delta t)} \sum_{t=1}^{T-\Delta t} \left[ \mu(t) - \langle \mu(t) \rangle \right] \left[ \mu(t + \Delta t) - \langle \mu(t) \rangle \right],$$

(A1)

where $T$ is the length of the record $\mu(t)$, $\langle \mu(t) \rangle$ its average, and $\sigma_{\mu}$ its standard deviation. For uncorrelated values of $\mu(t)$, $C(\Delta t)$ is zero for $\Delta t > 0$, because on average positive and negative products will cancel each other out. In the case of short-term correlations $C(\Delta t)$ has a characteristic decay time $\Delta t_x$. A prominent example is the exponential decay $C(\Delta t) \sim \exp(-\Delta t/\Delta t_x)$. Long-term correlations are described by a slower decay, e.g. diverging $\Delta t_x$, namely

$$C(\Delta t) \sim (\Delta t)^{-\gamma},$$

(A2)

with the correlation exponent $0 < \gamma < 1$.

Detrended Fluctuation Analysis (DFA) is a well studied method to quantify long-term correlations in the presence of non-stationarities \cite{2, 7, 14, 90–94}. The analysis of a considered record $\mu(t)$ of length $T$ consists of 5 steps \cite{14}:

1. Calculate the cumulative sum, the so-called profile:

$$Y(t) = \sum_{t'=1}^{t} (\mu(t') - \langle \mu(t) \rangle).$$

(A3)

2. Separate the profile $Y(t)$ into $T_{\Delta t} = \text{int}\frac{T}{\Delta t}$ segments of length $\Delta t$. Often, the length of the record is not a multiple of $\Delta t$. In order not to disregard information, the segmentation procedure is repeated starting from the end of the record and one obtains $2T_{\Delta t}$ segments.

3. Locally detrend each segment $\nu$ by determining best polynomial fits $p_{\nu}^{(n)}(t)$ of order $n$ and subsequently subtract it from the profile:

$$Y_{\Delta t}(t) = Y(t) - p_{\nu}^{(n)}(t).$$

(A4)

4. Calculate for each segment the variance (squared residuals) of the detrended $Y_{\Delta t}(t)$

$$F^2_{\Delta t}(\nu) = \frac{1}{\Delta t} \sum_{j=1}^{\Delta t} (Y_{\Delta t}^2[(\nu - 1)\Delta t + j])$$

by averaging over all values in the corresponding $\nu$th segment.

5. The DFA fluctuation function is given by the square-root of the average over all segments:

$$F(\Delta t) = \left[ \frac{1}{2T_{\Delta t}} \sum_{\nu=1}^{2T_{\Delta t}} F^2_{\Delta t}(\nu) \right]^{1/2}.$$  

(A6)

The averaging of $F^2_{\Delta t}(\nu)$ is additionally performed over members of similar activity level $M$.

If the record $\mu(t)$ is long-term correlated according to a power-law decaying auto-correlation function, Eq. (A2), then $F(\Delta t)$ increases for large scales $\Delta t$ also as a power-law:

$$F(\Delta t) \sim (\Delta t)^H,$$

(A7)

where the fluctuation exponent $H$ is analogous to the well-known Hurst exponent \cite{54, 93, 90}. The exponents are related via

$$H = 1 - \gamma/2, \quad \gamma = 2 - 2H.$$  

(A8)
When $\gamma = 1$ then $H_{\text{rand}} = 1/2$, that is the case of uncorrelated dynamics. If the correlations decay faster than $\gamma > 1$ then the random exponent $H_{\text{rand}} = 1/2$ is still recovered. Long-term correlations imply $0 < \gamma < 1$ and $1/2 < H < 1$. In practice, one plots $F(\Delta t)$ versus $\Delta t$ in double-logarithmic representation, determines the exponent $H$ on large scales and quantifies the correlation exponent $\gamma$. The order of the polynomials $p_\nu^{(n)}$ determines the detrending technique which is named DFA$n$, DFA0 for constant detrending, DFA1 for linear, DFA2 for parabolic, etc.

The subtraction of the average in Eq. (A3) is only necessary for DFA0. By definition the corresponding fluctuation function is only given for $\Delta t \geq n + 2$. The detrending order determines the capability of detrending. Since the local trends are subtracted from the profile, only trends of order $n - 1$ are subtracted from the original record $\mu(t)$. Throughout the paper we show the results using DFA2 which we found to be sufficient in terms of detrending.

Since the fluctuation functions $F(\Delta t)$ for single users are very noisy, it is useful to average fluctuation functions among various members. Thus, we first group the members in logarithmic bins according to their activity level, the total number of messages $M$ sent. Namely, we group all members that send 1-2, 3-7, 8-20, . . . messages in the period of data acquisition by using bins determined by $b = \text{int}(\ln M)$. Next we average the fluctuation function in the squares among all members from each group $b$ and obtain for every activity level of the members one DFA fluctuation function. The error bars, such as in Fig. 4, were obtained by subdividing each group and determining the standard deviations of the fluctuation exponents from different groups of the same activity level.

**Appendix B: Relation between the fluctuation exponent $H_\delta$ and the inter-event time distribution exponent $\delta$**

Following [9, 13, 28, 43, 46] we consider a time series which is equal to zero everywhere except at points $t_1, t_2, . . . , t_n, . . .$, at which it is equal to unity. If these points are taken from a fractal set with fractal dimension $d_t$, it can be shown [97] that the auto-correlation function, Eq. (A1), decreases for $t \rightarrow \infty$ as $C(t) \sim t^{-\gamma}$, Eq. (A2), where $\gamma = 1 - d_t$ and the power spectrum $S(f)$ of this time series diverges for $f \rightarrow \infty$ as $S(f) \sim f^{-\epsilon}$, with $\epsilon = 1 - \gamma = d_t$.

In the more general case, the distribution of the intervals $dt_n = t_n - t_{n-1}$ can decay as a power-law $p(dt) \sim (dt)^{-\delta}$, Eq. (1), where $1 < \delta < 3$. For $\delta > 2$, the set $t_1, t_2, . . . , t_n$ is a fractal, and the power spectrum decreases as a power-law $S(f) \sim f^{-d_t} = f^{-\delta + 1}$. For $2 < \delta < 3$, the set $t_1, t_2, . . . , t_n$ has a finite density, with $d_t = D = 1$, where $D$ is the dimension of the substrate. However, the power spectrum and the correlation function maintain their power-law behavior for $\delta < 3$. This indicated that although the time series itself is uniform, the temporal fluctuations remain fractal. In this case the exponent $\epsilon$ characterizing the low frequency limit is given by $\epsilon = 3 - \delta$. The maximal value of $\epsilon = 1$ is achieved when $\delta = 2$, which is called $1/f$-noise. If $\delta > 3$, $\epsilon = 0$ in the limit of low frequencies and white noise is recovered. In summary, we have

$$\epsilon = \begin{cases} 
\delta - 1 & \text{for } 1 < \delta < 2 \\
3 - \delta & \text{for } 2 < \delta < 3 \\
0 & \text{else}
\end{cases} ,$$  \hspace{1cm} (B1)

and with $H = \frac{1}{2}(\epsilon + 1)$ [14, 98]

$$H_\delta = \begin{cases} 
\delta/2 & \text{for } 1 < \delta < 2 \\
2 - \delta/2 & \text{for } 2 < \delta < 3 \\
1/2 & \text{else}
\end{cases} .$$  \hspace{1cm} (B2)

**Appendix C: Derivation of $\beta = 1 - H$**

We follow the considerations in [1]. To relate the exponent from Eq. (9), $\beta$, to the temporal correlation exponent $\gamma$, from Eq. (A2), and therefore to $H$, one can first rewrite Eq. (9) as:

$$r = \ln \frac{m_1}{m_0}$$  \hspace{1cm} (C1)

$$= \ln \frac{m_0 + \Delta m}{m_0} \quad \text{with } \Delta m = m_1 - m_0$$  \hspace{1cm} (C2)

$$= \ln \left( \frac{\Delta m}{m_0} + 1 \right)$$  \hspace{1cm} (C3)

$$\approx \frac{\Delta m}{m_0} \quad \text{for small } \frac{\Delta m}{m_0}$$  \hspace{1cm} (C4)
Next, the total increment of messages $\Delta m$ is expressed in terms of smaller increments $\mu(t)$, such as messages per day:

$$ \Delta m = \sum_{t=t_0+1}^{t_0+\Delta t} \mu(t), \quad (C5) $$

which is (assuming stationarity) statistically equivalent to

$$ \Delta m = \sum_{i=1}^{\Delta t} \mu(t) , \quad (C6) $$

and one can write

$$ r \approx \frac{1}{m_0} \sum_{i=1}^{\Delta t} \mu(t) \quad (C7) $$

for the growth rate. The conditional average growth is then

$$ \langle r(m_0) \rangle = \langle \frac{1}{m_0} \sum_{i=1}^{\Delta t} \mu(t) \rangle \quad (C8) $$

$$ \approx \frac{1}{m_0} \sum_{i=1}^{\Delta t} \langle \mu(t) \rangle . \quad (C9) $$

In order to obtain the conditional standard deviation

$$ \sigma(m_0) = \sqrt{\langle [r(m_0) - \langle r(m_0) \rangle]^2 \rangle}, \quad (C10) $$

one can continue

$$ r(m_0) - \langle r(m_0) \rangle = \frac{1}{m_0} \left( \sum_{i=1}^{\Delta t} \mu(t) - \sum_{i=1}^{\Delta t} \langle \mu(t) \rangle \right) \quad (C11) $$

$$ [r(m_0) - \langle r(m_0) \rangle]^2 = \frac{1}{m_0^2} \left( \sum_{i=1}^{\Delta t} (\mu(t) - \langle \mu(t) \rangle) \right)^2 \quad (C12) $$

$$ \langle [r(m_0) - \langle r(m_0) \rangle]^2 \rangle \approx \frac{1}{m_0^2} \sum_i \sum_j \sigma_\mu^2 C(i-j) , \quad (C13) $$

where $C(\Delta t)$ is the auto-correlation function of $\mu(t)$, Eq. (A1), and $\sigma_\mu$ is the standard deviation of $\mu(t)$. For long-term correlations it asymptotically decays as a power-law $C(\Delta t) \sim (\Delta t)^{-\gamma}$, Eq. (A2). Approximating the double sum with integrals, one obtains

$$ \langle [r(m_0) - \langle r(m_0) \rangle]^2 \rangle \approx \frac{1}{m_0^2} \sigma_\mu^2 \int_1^{\Delta t} \int_1^{\Delta t} (j-i)^{-\gamma} \, dj \, di \quad (C14) $$

$$ \sim \frac{1}{m_0^2} \sigma_\mu^2 (\Delta t)^{2-\gamma} , \quad (C15) $$

see also [14].

In order to relate $\Delta t$ and $m_0$, one can use

$$ \Delta t = x t_0 , \quad (C16) $$

where $x$ is an arbitrary (small) constant, that simply states how large $\Delta t$ is compared to $t_0$, and

$$ m_0 \sim t_0 , \quad (C17) $$
which states that the number of messages is proportional to time assuming stationary activity. Using these two arguments we obtain:

\[
\langle (r(m_0) - \langle r(m_0) \rangle)^2 \rangle \approx \frac{1}{m_0^2} \sigma_0^2 (x)^{2-\gamma} (t_0)^{2-\gamma}
\]

(C18)

\[
\sim \sigma_\mu m_0^{-\gamma}
\]

(C19)

\[
\sigma(m_0) \sim \sigma_\mu m_0^{-\gamma/2}.
\]

(C20)

Comparing with Eq. (19) we finally get

\[
\beta = \gamma/2,
\]

(C21)

and with Eq. (A8), \(\gamma = 2 - 2H\):

\[
\beta = 1 - H.
\]

(C22)

Thus, \(\beta_{\text{rand}} = 1/2\) implies \(H_{\text{rand}} = 1/2\) and Gibrat’s law with \(\beta_G = 0\) is obtained for \(H_G = 1\) and \(\gamma_G = 0\). Any value \(0 < \beta < 1/2\) implies correlations with \(1/2 < H < 1\).

Appendix D: Derivation of \(\beta_{\text{BA}} = 1/2\)

We follow the considerations in [1]. The BA network model (also known as Barabasi Albert Model) has been described analytically. In particular, it has been shown that each nodes’ degree increases as

\[
k(t) \sim \left( \frac{t}{t^*} \right)^b,
\]

(D1)

where \(t^*\) is the time when the corresponding node was introduced to the system and \(b\) is the dynamics exponent in growing network models \((b = 1/2\) for the standard BA) [80]. Accordingly, here the growth rate, Eq. (12), is \(r_{\text{BA}} = \frac{1}{2} \ln \frac{t}{t_0}\), which we also find in Fig. 23.

To obtain \(\sigma_{\text{BA}}(k_0)\) one can use analogous considerations as for \(\sigma(m_0)\) before. Due to Eq. (12) here we have

\[
r_{\text{BA}} \approx \frac{1}{k_0} \sum_{i=1}^{\Delta t} \kappa(t),
\]

(D2)

where \(\kappa(t)\) are small increments analogous to \(\mu(t)\), whereas Eq. (D1) implies

\[
\kappa(t) \sim (\Delta t)^{-1/2}
\]

(D3)

for the standard BA model. As before, the conditional standard deviation of the growth rate is

\[
\langle [r_{\text{BA}}(k_0) - \langle r_{\text{BA}}(k_0) \rangle]^2 \rangle \approx \frac{1}{k_0^2} \sum_{i=1}^{\Delta t} \sum_{j=1}^{\Delta t} \sigma_\kappa^2 C(j - i).
\]

(D4)

In the uncorrelated case \(C(j - i) = \delta_{ij}\), the double sum can be reduced to a single one:

\[
\sigma_{\text{BA}}^2(k_0) = \frac{1}{k_0^2} \sum_{i=1}^{\Delta t} \sigma_\kappa^2(i).
\]

(D5)

As shown below, \(\sigma_\kappa(i) \sim i^{-1/4}\), and integration leads to

\[
\sigma_{\text{BA}}^2(k_0) \sim \frac{1}{k_0^2} \int_{\Delta t} \left( i^{-1/2} \right) di
\]

\[
\sim \frac{1}{k_0^2} (\Delta t)^{1/2}.
\]

(D6)
Eliminating $\Delta t$ using $k \sim t^{1/2}$, Eq. (D1) with $b = 1/2$, one obtains
\[ \sigma_{BA}(k_0) \sim k_0^{-1/2}. \] (D8)
That is, we obtain $\beta_{BA} = 1/2$ as found numerically.

Remains to show $\sigma_\kappa(t) \sim t^{-1/4}$. We assume new links are set according to a Poisson process, whereas every new link of a node represents an event. The intervals between these events (asymptotically) follow an exponential distribution $p(\tau) = \lambda e^{-\lambda \tau}$. Accordingly, $\kappa(t)$ is a sequence of zeros and only one when a new link is set to the corresponding node. The standard deviation of this sequence is
\[ \sigma_\kappa \sim \lambda^{1/2}. \] (D9)
Due to Eq. (D1) the rate parameter decreases like
\[ \lambda(t) \sim t^{-1/2}. \] (D10)
Accordingly,
\[ \sigma_\kappa(t) \sim t^{-1/4}. \] (D11)

As mentioned in Sec. III C 3, a fitness model has been introduced (81) taking into account different fitnesses of the nodes of acquiring links and therefore involving a distribution of $b$-exponents. The spread of growth rates $r$ could be related to the distribution of fitness. On the other hand, the growth according to Eq. (D1) is superimposed with random fluctuations that we characterize with the exponent $\beta$.

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