Correlated network of networks enhances robustness against catastrophic failures

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Abstract

Networks in nature rarely function in isolation but instead interact with one another with a form of network of networks (NoN). Network of networks with interdependency between distinct networks contains instabilities of abrupt collapse related to the global rule of activation. As a remedy of the collapse instability, here we investigate a model of correlated NoN and find that the collapse instabilities can be removed with a specific pattern of correlated connectivity. We find that when hubs provide majority of interconnections and interconnections are convergent, a system of networks becomes stable systematically. Our study identifies a stable structure of correlated NoN against catastrophic failures and suggests a plausible way to enhance network robustness by manipulating connection patterns.

Keywords: Complex network, Network of network, Network robustness

1. Introduction

Real-world complex systems ranging from critical infrastructure \cite{1, 2, 3} and transportation networks \cite{4, 5} to living organisms \cite{6, 7, 8} are rarely formed by an isolated network but by a network of networks (NoN) \cite{3, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23}. For instance, different kinds of critical infrastructures such as power grid and Internet are coupled and

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interact with one another [1, 2]. In addition, many living systems including brain networks [6, 24] and cellular networks [7] consist of different modules strongly connected and interconnections between them.

Several models of a system of networks have been proposed with the role of interconnections that are links across different networks [9, 3, 10]. Models of NoN introduced fall into three classes according to the functionality of the interconnections: Modular NoN (M-NoN), Catastrophic NoN (C-NoN), and Robust NoN (R-NoN). A primitive model of NoN is Modular NoN in which intraconnection within a network and interconnections between different networks have no difference in function [9]. Thus, this model essentially corresponds to a single modular network with different density of intra- and interconnections since nodes between different networks do not control each other.

However, considering distinct nature of intra and interconnections in NoN, a different role for different types of connections may be required. For example, when different networks function interdependently, interconnections should not play the same role as intraconnection but control the state of a connected node in the other networks [3, 10]. In C-NoN model, the state of a node is determined by the global characteristics of an interconnected node in a different network [3, 25]. To be specific, a node can be active only if interconnected nodes in a different network belong to the global giant component. Such global rule results in an extreme instability of a system of networks since a small perturbation can trigger catastrophic collapse of the whole giant component.

In order to resolve the conflict between the extreme fragility and robust systems of networks observed widely in reality such as the brain, R-NoN model in which the state of a node is controlled by local property of interconnected nodes have been proposed [10, 11]. Thus, in R-NoN model, a node can be active even though interconnected nodes in a different network do not belong to the global giant component. With this modification, R-NoN still maintains the functionality of control across different networks but becomes robust.

Beside R-NoN, it is still of interest how to produce more robust C-NoN because there are examples to follow the global rule such as power-grid. Catastrophic NoN (C-NoN) model involves vulnerability related to the global rule leading to potential danger of abrupt collapse. Thus, here we investigate a modified model taking into account correlation of the patterns of connectivity in NoN as a remedy of the collapse instabilities. So far, the majority
of research about networks of networks have studied NoN with uncorrelated and one-to-one interconnections [3, 25]. In contrast, a system of coupled networks in reality are composed with one-to-many interconnections and degree-degree correlation between nodes in distinct networks [5, 6, 26, 27, 28]. For instance, for the case of human brain networks, non-trivial patterns of connections between networks have been reported for resting state and in task [6]. Correlated coupling was also observed in several different types of complex systems such as transportation networks [26], social networks [27], and critical infrastructure networks [2, 28].

In this study we find that the collapse instabilities in C-NoN can be removed turning the model stable by introducing correlated NoN. Specifically, we investigate the effect of degree-degree correlation on network robustness under random removal of nodes by extending the previous analysis [6]. We find that when hubs are major source of outgoing links and the interconnections are convergent between hubs, NoN becomes stable to function properly. Our study provides an optimal design of correlated NoN against external perturbation and also be a possible answer of stable functioning of correlated networks in reality.

2. Model and theory

We consider a network of networks composed of two networks, $A$ and $B$, with interconnections between networks, for the sake of simplicity. Each node in NoN can have two different types of links, In- and out-links. In-link refers the connections inside the same network while out-link is the connections between nodes on different networks.

Here, we examine two different modes of interactions of out-links [6]: Catastrophic NoN and Modular NoN. C-NoN represents that a node in the network $A$ operates properly only when one of the reciprocal nodes in the network $B$ connected by out-links also functions properly. In other words, a node in the network $A$ cannot be active either it does not belong to the giant component on the network $A$ or it loses all connectivity to the network $B$. On the other hand, for M-NoN, a node in the network $A$ can be active if it belongs to the giant component through either in-link or out-link. Thus, even though node $i$ in $A$ is completely decoupled from the network $B$, node $i$ can be active as long as it still belongs to the giant component of $A$ via in-link. Therefore, for M-NoN mode of interactions, there is no cascading after removing a fraction of nodes.
In order to assess the robustness of the system against random failure of a fraction $p$ of nodes, we measure the size of giant component after removing nodes. We also identify the percolation threshold $p_c$ at which the giant component disappears, to measure stability of NoN. NoN with low $p_c$ corresponds to stable structures because a lot of nodes need to be removed to break it down, whereas high $p_c$ represents vulnerable structures.

2.1. Catastrophic network of networks

In this section, we introduce theory for C-NoN mode of interactions to find the size of giant component and percolation threshold. Initially, all nodes in both networks A and B are active. A fraction $p_A$ and $p_B$ of nodes randomly chosen are removed from the networks A and B, respectively. Then, a node is active only if it belongs to the giant component on its network via in-links and at the same time connects to the giant component on the other network via one of its out-links. Nodes that do not satisfy the survival condition are removed from the network iteratively. Note that nodes that do not have any out-links at the beginning can be active as long as they remain to connect with the giant component via in-links.

To obtain the percolation threshold $p_c$, we introduce the joint degree distributions of in- and out-degree as $P(\vec{k})$ where $\vec{k} = (k_{in}^A, k_{in}^B, k_{out}^A, k_{out}^B)$. We also introduce the conditional degree distribution for a pair of connected nodes on different networks to take into account degree-degree correlation, $P_{AB}(k_{in}^A|k_{in}^B)$ and $P_{BA}(k_{in}^B|k_{in}^A)$. Next, we develop a framework for the robustness of NoN on the locally tree-like structure with arbitrary joint degree distribution and conditional degree distribution [6].

We define $u_A$ and $u_B$ respectively as the probability that a node on networks A and B reached by a randomly chosen in-link does not belong to the mutually connected giant component. $u_A$ and $u_B$ can be expressed by the following self-consistency equation

\[
1 - u_i = p_i \left[ \sum_{k} \frac{k_{in}^i P(\vec{k})}{\langle k_{in}^i \rangle} (1 - u_i^{k_{in}^i - 1}) (\delta_{k_{out}^i,0} + 1 - w_{k_{out}^i}^{k_{in}^i}) \right], \tag{1}
\]

where $i \in \{A, B\}$ and $\delta_{i,j}$ is the Kronecker delta. Here, $w_{k_{in}^i}$ is the probability that a node reached by a randomly chosen out-link from a node on the network $i$ with indegree $k_{in}^i$ does not belong to the giant component of the different network. The first term $(1 - u_i^{k_{in}^i - 1})$ represents the probability that a node with $k_{in}^i$ belongs to the giant component in the network $i$, and the
second term represents that the probability that a node with \( k^i_{\text{in}} \) connects with the giant component of the different network through out-links. By the term \( \delta_{k^i_{\text{out}},0} \) in the Eq. (1), a node without out-links \( (k^i_{\text{out}} = 0) \) can be treated differently with other nodes \( (k^i_{\text{out}} \neq 0) \). Then, the probability \( w_{k^i_{\text{in}}} \) can be expressed as

\[
1 - w_{k^i_{\text{in}}} = p_i \left[ 1 - \sum_{k^i_{\text{in}}} P(k^j_{\text{in}}|k^i_{\text{in}}) u_{j,k^j_{\text{in}}}^{k^j_{\text{in}}} \right].
\]  

(2)

Obtaining \( u_i \) and \( w_{k^i_{\text{in}}} \) by solving these equations, the size \( G_i \) of the mutually connected giant component of C-NoN is given by

\[
G_i = p_i \left[ \sum_k P(k) (1 - u_k^{k_{\text{in}}})(\delta_{k^i_{\text{out}},0} + 1 - w_{k^i_{\text{in}}}^{k_{\text{out}}}) \right].
\]  

(3)

2.2. Modular network of networks

For the M-NoN, a node can survive if it belongs to at least one of the giant component of both networks. Given degree distributions, the probability \( \nu_i \) that a node reached by a randomly chosen in-link of network \( i \) does not belong to the giant component of M-NoN is given by

\[
1 - \nu_i = p_i \left[ \sum_k \frac{k_{\text{in}}^i P(k)}{\langle k_{\text{in}}^i \rangle} (1 - \nu_k^{k_{\text{in}}^i - 1} \mu_{k_{\text{in}}^i}) \right].
\]  

(4)

Here, \( \mu_{k_{\text{in}}^i} \) is the probability that a node reached by a randomly chosen out-link from a node on the network \( i \) with indegree \( k_{\text{in}}^i \) does not belong to the giant component of the different network. And, the probability \( \mu_{k_{\text{in}}^i} \) can be obtained by following,

\[
1 - \mu_{k_{\text{in}}^i} = p_i \left[ 1 - \sum_{k_{\text{in}}^i} P(k_{\text{in}}^j|k_{\text{in}}^i) \nu_{j,k_{\text{in}}^j}^{k_{\text{in}}^j} \right].
\]  

(5)

For M-NoN, a node in the network \( i \) can survive either it belongs to the giant component on network \( i \) or to the giant component on network \( j \) by
Figure 1: Diagram of correlated network of networks according to the parameters $\alpha$ and $\beta$. Hubs (red nodes) and non-hubs (blue nodes) can have both in-links (solid lines) and out-links (dotted lines). When $\alpha > 0$, the out-links are more likely to be found attached to hub nodes whereas when $\alpha < 0$, non-hub nodes are more likely to have out-links. When $\beta > 0$, hubs prefer to connect with hubs in a different network but when $\beta < 0$, hubs in one network prefer to connect to non-hub nodes in a different network.

Connecting out-links. Once we obtain $\nu_i$ and $\mu_{k_{in}}$, the size $\mathbb{G}_i$ of the giant component of $\mathbb{M}$ is

$$G_i = p_i \left[ \sum_{\vec{k}} P(\vec{k}) \left( 1 - \nu_i^{k_{in}} \mu_{k_{out}}^{k_{out}} \right) \right]. \quad (6)$$

2.3. Correlation in network of networks

In real-world complex systems, NoN are not made randomly but with a certain degree-degree correlation. Correlated coupling is observed in several different kinds of complex systems such as transportation networks [26], social networks [27], and critical infrastructure networks [2, 28], and crucial for structural and dynamical properties of networks [29, 30, 31]. For instance, functional brain networks of the human show a peculiar correlation pattern [6]. In this paper, we consider the degree-degree correlation using two scaling parameters, $\alpha$ and $\beta$ (Fig. 1) as observed in functional networks of the human...
brain [6]. The parameter $\alpha$ is defined as

$$k_{out} \sim k_{in}^{\alpha}.$$  \hfill (7)

Thus, for $\alpha > 0$ hubs of each network also have many out-links, whereas for $\alpha < 0$ nodes with low degree have many out-links (Fig. 1). The other parameter $\beta$ is defined as

$$k_{in}^{nn} \sim k_{in}^{\beta},$$  \hfill (8)

where $k_{in}^{nn}$ is the average indegree of the nearest neighbors in the other network. Therefore, $\beta$ quantifies indegree-indegree correlation between two connected nodes by interconnections. For $\beta > 0$, hubs connect with other hubs in the different network. Instead for $\beta < 0$, hubs in a network connect with nodes with less degree in the other network (Fig. 1). Note that uncorrelated NoN corresponds to $\alpha = 0$ and $\beta = 0$.

3. Result

3.1. Effect of density of out-links in NoN

We first examine the robustness of NoN by changing the density of links in order to check the effect of out-links. As an instructive example, we consider two coupled Erdős-Rényi (ER) networks. For the ER NoN with no degree correlation, joint degree distribution can be factorized as $P(k) = P_{in}(k_{in})P_{out}(k_{out})$ and conditional degree distribution can be simply expressed as $P(k_{in}|k_{out}) = P_{in}(k_{in})$. When two networks have the same average indegree, $\langle k_{in} \rangle = \langle k_{out} \rangle = \langle k_{in} \rangle$, and the fraction of removed nodes are the same for both networks, $p_A = p_B = p$, Eqs. (1) and (2) can be simply reduced into a single equation:

$$u = 1 - p \left[ 1 - e^{\langle k_{in} \rangle(u-1)} \right] \left[ e^{-\langle k_{out} \rangle} + 1 - e^{p\langle k_{out} \rangle(e^{\langle k_{in} \rangle(u-1)}-1)} \right].$$  \hfill (9)

where $\langle k_{out} \rangle$ is the average out-degree. Once we define the function

$$f(u) = u - 1 + p \left[ 1 - e^{\langle k_{in} \rangle(u-1)} \right] \left[ e^{-\langle k_{out} \rangle} + 1 - e^{p\langle k_{out} \rangle(e^{\langle k_{in} \rangle(u-1)}-1)} \right],$$  \hfill (10)

one can obtain the percolation threshold $p_c$ by imposing the conditions $f(u) = f'(u) = 0$. In addition, tricritical line $(\langle k_{in} \rangle, \langle k_{out} \rangle, p)$ between continuous and discontinuous transitions can be computed by the conditions $f(u) = f'(u) = f''(u) = 0$. 

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Figure 2: (a) Percolation threshold $p_c$ of C-NoN with two ER networks, predicted by theory. For high $\langle k_{out}\rangle$ and low $\langle k_{in}\rangle$, NoN is stable to maintain mutual connectivity under random removals of nodes. (b) The size of jump at the percolation threshold of C- NoN. While transition undergoes second order for small $\langle k_{out}\rangle$, it becomes discontinuous as $\langle k_{out}\rangle$ increases. (c) Percolation threshold $p_c$ of M-NoN for ER NoN with no degree correlation. NoN becomes more stable with increasing either $\langle k_{in}\rangle$ or $\langle k_{in}\rangle$. (d) The size of giant component for both C-NoN (open symbol) and M-NoN (filled symbol) modes of interactions as a function of a fraction of removed nodes $p$. Analytic calculation (line) and numerical simulation (symbols) are shown together.
For M-NoN, the self-consistency equation is similarly given by

\[
1 - \nu = p \left[ 1 - e^{(k_{in})(\nu-1)} e^{(k_{out}) p e^{(k_{in})(\nu-1)-1}} \right].
\] (11)

Then, one can obtain the percolation threshold with the conditions \(g(\nu) = g'(\nu) = 0\), if we define

\[
g(\nu) = \nu - 1 + p \left[ 1 - e^{(k_{in})(\nu-1)} e^{(k_{out}) p e^{(k_{in})(\nu-1)-1}} \right].
\] (12)

Note that the percolation transition of M-NoN is always second order and hence a tricritical line does not exist.

Increasing the density of out-links, NoN with catastrophic interactions becomes getting vulnerable as depicted in Fig. 2(a). In addition, the transition becomes discontinuous above tricritical line and the size of the jump at transition increases as well with increasing \(\langle k_{out} \rangle\) [Fig. 2(b)]. For C-NoN, out-links force interconnected systems to be more vulnerable and prone to abrupt collapse due to cascading failures. On the other hand, in-links preserve the connectivity and produce more robust structures. In conclusion, NoN with high \(\langle k_{in} \rangle\) and low \(\langle k_{out} \rangle\) shows stable structure for C-NoN.

For M-NoN, however, out-links play the opposite role. High density of out-links enhances the network robustness by adding potential detour of connectivity [Fig. 2(c)]. For M-NoN, out-links contribute to maintain robustness of networks but for C-NoN out-links can cause the opposite effect. Thus, the optimal design of the connections between networks is called for maintaining stable functioning for both M-NoN and C-NoN.

3.2. Generating correlated networks of networks

In order to examine the effect of degree-degree correlation, we first construct NoN with correlation \((\alpha, \beta)\). We construct a network drawn from an indegree distribution \(P_i(k_{in})\), by following the configuration model. Next, the stubs of outgoing links are assigned to each node with the probability proportional to \(k_{in}^\alpha\). Connecting two nodes in different networks with a relationship \(k_{in}^{\alpha n} \sim k_{in}^{\beta n}\) is non-trivial. We cannot simply assign a set of connections for out-links from the joint distribution \(P(\vec{k})\) since such a set almost certainly fails to satisfy the topological constraint because of the reciprocal relation between \(k_{in}^{\alpha n} \sim k_{in}^{\beta n}\). We choose randomly node \(i\) in the network \(A\) if it has
Figure 3: (a) Percolation threshold and (b) size of jump of C-NoN in correlated ER NoN with $N = 10^4$, $\langle k_{in} \rangle = 2$, and $\langle k_{out} \rangle = 1$ for different $\alpha$ and $\beta$. When $\alpha \approx -1$ or $\alpha > 0.5$ and $\beta > 0$, NoN becomes stable against random failure. In contrast, when $-0.5 < \alpha < 0.5$ and $\beta < 0$, NoN is vulnerable to catastrophic collapse. (c) percolation threshold of M-NoN with correlated ER NoN with the same parameters as C-NoN. High $\alpha$ and $\beta$ region is robust against random failure for M-NoN. (d) $\beta_{gen}$ observed from realized networks at a given $(\alpha, \beta)$. The value $\beta_{gen}$ is obtained by linear regression.
available outlinks. Next, we connect the node \( i \) with a node \( j \) with the degree \( k_{in}^B \) in the network \( B \) with the probability that follows a Poisson distribution \( P(k_{in}^B) \) with the mean value \( \lambda = \langle C_\beta k_{in}^B \rangle \) where \( C_\beta = k_{max}^{(1-\beta)/2} \). This processes repeat until there are no out-links left. This algorithm cannot make NoN with exactly corresponding \( \beta \) for most sets of \((\alpha, \beta)\), but it can guarantee that numerically generated \( \beta_{gen} \) increases or decreases in a monotonic manner with changing \( \beta \) [Fig. 3(d) and 4(d)].

3.3. Robustness of correlated networks of networks

To search robust structure of correlated NoN, we generate a network with the above algorithm and obtain joint and conditional degree distributions from the realized networks with \((\alpha, \beta)\) for applying the theory. Next, we identify the critical fraction \( p_c \) of nodes removal by imposing the condition \( G(p_c) = 0 \), showing the network robustness with a given correlation. In order to examine the effect of the correlated structure of NoN, we calculate \( p_c(\alpha, \beta) \) for the both modes of C-NoN and M-NoN with ER networks and scale-free (SF) networks. The small \( p_c(\alpha, \beta) \) represents robust structure against an external perturbation.

For ER network, when \( \alpha \approx -1 \), low \( p_c \) is observed regardless of \( \beta \), indicating stable NoN [Fig. 3(a)]. In this region, hubs are isolated in a single network and maintain effectively the giant component. As a result, the extensive size of jump at \( p_c \) vanishes [Fig. 3(b)]. Another stable region is located at \( \alpha > 0 \) and \( \beta > 0 \). High \( \alpha \) and \( \beta \) guarantees that a lot of hub-hub interconnections, so that hubs are more likely protected from cascading failures. When \(-0.5 < \alpha < 0.5 \) and \( \beta < 0 \), a system of networks is highly vulnerable to catastrophic cascading. With these parameters, hubs connect to nodes with less degree nodes in the other network, leading to that hubs can be easily attacked by interdependency. For M-NoN, the network robustness enhances with increasing \( \alpha \) and \( \beta \) monotonically [Fig. 3(c)]. When \( \alpha > 0 \) and \( \beta > 0 \), both in- and out-links converge toward hubs and the giant component can be preserved with only a few hubs of each network. Therefore, for M-NoN, high \( \alpha \) and \( \beta \) region is robust against random failure.

The impact of correlation is more clear in SF networks because of a key role of hubs with inhomogeneous degree distributions. When \( \alpha < 0 \) for C-NoN, a networked system is stable (low \( p_c \)) because hubs are protected from cascading failures [Fig. 4(a)]. When \( \alpha > 0.5 \) and \( \beta > 0 \), networks are also stable since hubs is more likely active due to a lot of interconnections between them. However, for intermediate \( \alpha \) (0 < \( \alpha < 0.5 \)) and divergent
interconnections ($\beta < 0$), hubs are easily exposed to cascading failures since they connect to non-hub nodes in the other network. Therefore, in this region, the NoN is fragile to random attack and results in abrupt collapse as shown in Fig. 4(b). For M-NoN, NoN is more vulnerable when $\alpha < 0$ because hubs have only few out-links as in ER networks [Fig. 4(c)].

In conclusion, degree-degree correlation in NoN allows us to find a stable structure for functioning of NoN. When hubs have many interconnections ($\alpha \approx 1$) and hub-hub interconnections are abundant ($\beta > 0$), NoN can maintain robust structure for both C-NoN and M-NoN. In contrast, when $\alpha < 0$, NoN is vulnerable for M-NoN and NoN is at risk of catastrophic collapse when $\beta < 0$ for C-NoN.
4. Discussion

We study the robustness of a system of networks with degree-degree correlation and one-to-many interconnections between distinct networks. We investigate the effect of degree-degree correlation on network robustness with different modes of interconnections. For uncorrelated NoN, out-links reduce network robustness for C-NoN while they enhance the robustness for M-NoN. However, taking into account the degree correlation, we find stable structure in correlated networks of network for both C-NoN and M-NoN. Specifically, when hubs provide most interconnections and the interconnections are convergent, networks of networks become more robust for both modes of interconnections. Our study of correlated NoN can shed light on the origin of reliable functioning of interconnected networks. In addition to correlated NoN, robust NoN model which is recently proposed [10, 11] can be another plausible solution of stable functioning of NoN and also allow us to find the core areas in NoN [10, 32, 33, 34, 35, 36].

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References


